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Electroelastic gap waves between dissimilar piezoelectric materials in different classes of symmetry

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ABSTRACT

Shear surface waves guided by a gap between dissimilar piezoelectric half-spaces that belong to different classes of material symmetry are systematically investigated. It is shown that at most two surface waves can be guided by the structure. Equations for calculating surface wave velocities of propagation and corresponding existence conditions are obtained in closed form. For some special cases, surface wave velocities are derived explicitly. It is shown that despite the presence of a characteristic length in the problem (that is, the gap width) a non-dispersive surface wave independent of the wavelength and gap width can be guided by the structure. Electroelastic wave components and existence conditions of this non-dispersive wave are obtained. It is demonstrated that both the gap width and the dielectric permittivity of the gap have significant influences on the dynamic behaviour of the structure so that even small gaps cannot always be simplified and modelled as permeable, impermeable or absorbent with infinitesimal width.

1. Introduction

Acoustic wave devices based on piezoelectric structures containing gaps have been extensively used for various purposes (Detaint et al., 1989; Tiersten, 1995; Yoshimoto et al., 1995). The wide application range of piezoelectric materials is based on their electromechanical sensitivity, service reliability and stability accompanied by electromechanical coupling and new possibilities for existence of surface waves. The studies of shear acoustic surface waves in piezoelectric materials were initiated by Bleustein (1968) and Gulyaev (1969) about 40 years ago. The pure shear acoustic surface wave they theoretically predicted has a simple electroelastic structure and has found many applications in different wave devices. Maerfeld and Tournois (1971) examined surface waves at the interface of dissimilar piezoelectric half-spaces with and without a conducting electrode embedded between them. Then, Danicki (1994) studied surface waves guided by an embedded grounded electrode in piezoelectric material, Liu et al. (2003) studied Bleustein–Gulyaev waves in prestressed layered piezoelectric structures, Danoyan and Piliposian (2007, 2008) considered surface waves in a piezoelectric half-space with hard and soft layers, Danoyan et al. (2009) studied surface gap waves in layered electro-magneto-elastic structures, Melkumyan (2007) and Wang et al. (2007) studied the existence of new surface waves when a coupling is present between electric, elastic and magnetic fields, Melkumyan and Mai (2008) studied the effect of imperfect bonding on interface waves guided by piezoelectric/piezomagnetic composites, Yang and Zhou (2005) analysed gap waves between a piezoceramic half-space and semiconductor layer, Gulyaev and Plesskii (1977) and Yang (2006) considered piezoelectric and piezoelectromagnetic gap waves, Syngellakis and Lee (1993) studied piezoelectric wave dispersion curves for infinite anisotropic plates and Li and Yang (2006) examined piezoelectric surface waves between a piezoelectric half-space and a plate. However, all these studies on piezoelectric gap waves either only discuss simple special cases or use numerical methods to obtain results for fixed materials. Further, they are limited to materials with hexagonal material symmetry only.

In the present paper, we present a systematic analytical and numerical study of surface waves guided by the gap between two piezoelectric materials which belong to the same or different classes of material symmetry. Existence conditions for all possible parameters of the structure are studied. For some special cases, explicit expressions are provided for surface wave velocities. It is shown that, despite the presence of a characteristic length in the structure (i.e., the gap width), a non-dispersive surface wave can be guided if the gap's dielectric permittivity is appropriately chosen. As small gaps are present in many structures, particularly crack faces are located at very small non-zero distance from each other, the case of a small gap will be considered separately. It is further shown that in dynamic problems modelling even very thin gaps as infinitesimal and using simple permeable, impermeable or

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absorbent idealized boundary conditions can lead to incorrect results.

2. Problem statement

Consider two dissimilar piezoelectric half-spaces with mechanically free surfaces located at a distance a from each other. Both piezoelectric materials belong to one of the following classes of material symmetry: 6, 4, 6 mm, 4 mm, 622, 422. Examples of piezoelectric materials of crystal classes 6, 4, 6 mm, 4 mm, 622, 422 include, respectively, lithium iodate $LiIO_3$, potassium strontium niobate $KSr_2Nb_5O_{15}$, cadmium sulphide CdS, barium titanate $BaTiO_3$, β-quartz SiO_2 and paratellurite TeO_2. Information on materials in these symmetry classes can be found in the IEEE Standard on Piezoelectricity (Meitzler et al., 1988). The gap between piezo-electric half-spaces is either a vacuum or filled by a gas (e.g., air) or occupied by a dielectric material with no acoustic contact with the piezoelectric half-spaces. A Cartesian coordinate system, xyz, is introduced in such a way that x- and z-axes belong to the free surface of one of the half-spaces. The piezoelectric materials are both poled along the z-direction and occupy the half-spaces $y > a$ and $y < 0$ (Fig. 1).

The interconnected dynamic electroelastic fields in piezoelectric materials can be described in the quasi-static approximation by the following governing equations:

$$\partial\sigma_{ij}/\partial x_j = \rho\partial^2 u_i/\partial t^2, \quad \partial D_i/\partial x_i = 0, \quad E_i = -\partial\varphi/\partial x_i, \tag{1}$$

geometric relations:

$$\gamma_{ij} = 0.5(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{2}$$

and constitutive relations:

$$\sigma_{ij} = C_{ijkm}\gamma_{km} - e_{kij}E_k, \quad D_i = e_{ikm}\gamma_{km} + \varepsilon_{ik}E_k, \tag{3}$$

where $\rho,\ \varphi,\ u_i$ are mass density, electric potential function, elastic displacement, γ_{km} and σ_{ij} are strain and stress tensors, D_i and E_k are electric displacement and electric field intensity, and $C_{ijkm}, e_{kij}, \varepsilon_{ik}$ are elastic, piezoelectric and dielectric coefficients. For compactness, the governing equations, geometric and constitutive relations in Eqs. (1)–(3) are written in tensor form with '1','2' and '3' in indices referring to the Cartesian axes x, y and z, respectively.

For piezoelectric materials under consideration, the constitutive relations (3) in the two index notation for the symmetry classes under consideration can be expressed as follows:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\\ D_x\\ D_y\\ D_z\end{bmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} & 0 & 0 & -e_{31}\\ c_{12} & c_{11} & c_{13} & 0 & 0 & -c_{16} & 0 & 0 & -e_{31}\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33}\\ 0 & 0 & 0 & c_{44} & 0 & 0 & -e_{14} & -e_{24} & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0 & -e_{15} & -e_{25} & 0\\ c_{16} & -c_{16} & 0 & 0 & 0 & c_{66} & 0 & 0 & 0\\ 0 & 0 & 0 & e_{14} & e_{15} & 0 & \varepsilon_{11} & 0 & 0\\ 0 & 0 & 0 & e_{24} & e_{25} & 0 & 0 & \varepsilon_{11} & 0\\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33}\end{pmatrix}\begin{bmatrix}\gamma_x\\ \gamma_y\\ \gamma_z\\ 2\gamma_{yz}\\ 2\gamma_{xz}\\ 2\gamma_{xy}\\ E_x\\ E_y\\ E_z\end{bmatrix}. \tag{4}$$

The following relations between the piezoelectric coefficients are valid in the corresponding classes of material symmetry:

$$e_{24} = e_{15}, \quad e_{25} = -e_{14} \quad \text{in classes 6 and 4;} \tag{5}$$

$$e_{14} = 0, \quad e_{25} = 0, \quad e_{24} = e_{15} \quad \text{in classes 6 mm and 4 mm;} \tag{6}$$

$$e_{15} = 0, \quad e_{24} = 0, \quad e_{25} = -e_{14} \quad \text{in classes 622 and 422.} \tag{7}$$

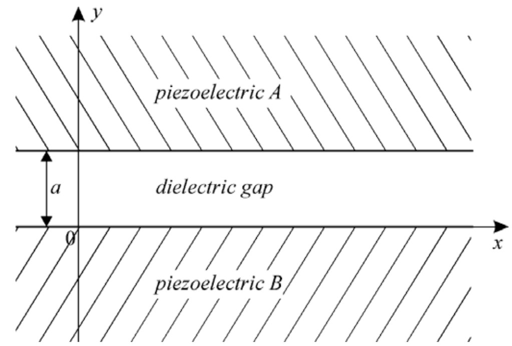

Fig. 1. Schematic illustration of dielectric gap and piezoelectric half-spaces.

Using Eqs. (4)–(7) the anti-plane problem for all the mentioned classes of material symmetry can be considered simultaneously (Danoyan and Piliposian, 2007, 2008). In the case of anti-plane motion:

$$u_x = u_y = 0, \quad u_z = w(x,y,t), \quad \varphi = \varphi(x,y,t). \tag{8}$$

From Eqs. (8), (1) and (2), it follows that the non-vanishing strain and electric field intensity components are:

$$\gamma_{xz} = 0.5\partial w/\partial x, \quad \gamma_{yz} = 0.5\partial w/\partial y, \quad E_x = -\partial\varphi/\partial x, \quad E_y = -\partial\varphi/\partial y. \tag{9}$$

Substituting Eqs. (5)–(9) into Eqs. (4) and (1), we obtain the following equations:

$$c_{44}\nabla^2 w + e_{15}\nabla^2\phi = \rho\partial^2 w/\partial t^2, \quad -\varepsilon_{11}\nabla^2\phi + e_{15}\nabla^2 w = 0, \tag{10}$$

and the following relations

$$\sigma_{xz} = c_{44}\frac{\partial w}{\partial x} + e_{15}\frac{\partial\varphi}{\partial x} - e_{14}\frac{\partial\varphi}{\partial y}, \quad \sigma_{yz} = c_{44}\frac{\partial w}{\partial y} + e_{15}\frac{\partial\varphi}{\partial y} + e_{14}\frac{\partial\varphi}{\partial x}, \tag{11}$$

$$D_x = -\varepsilon_{11}\frac{\partial\varphi}{\partial x} + e_{15}\frac{\partial w}{\partial x} + e_{14}\frac{\partial w}{\partial y}, \quad D_y = -\varepsilon_{11}\frac{\partial\varphi}{\partial y} + e_{15}\frac{\partial w}{\partial y} - e_{14}\frac{\partial w}{\partial x}. \tag{12}$$

Eq. (10) can be decoupled by introducing a pseudo-electric potential function:

$$\psi = \varphi - e_{15}\varepsilon_{11}^{-1}w. \tag{13}$$

Using Eq. (13), Eqs. (10)–(12) can be rewritten in the following form:

$$\nabla^2 w = s_{sh}^2\partial^2 w/\partial t^2, \quad \nabla^2\psi = 0 \tag{14}$$

and

$$\sigma_{xz} = \bar{c}_{44}\frac{\partial w}{\partial x} + e_{15}\frac{\partial\psi}{\partial x} - \frac{e_{14}e_{15}}{\varepsilon_{11}}\frac{\partial w}{\partial y} - e_{14}\frac{\partial\psi}{\partial y}, \tag{15}$$

$$\sigma_{yz} = \bar{c}_{44}\frac{\partial w}{\partial y} + e_{15}\frac{\partial\psi}{\partial y} + \frac{e_{14}e_{15}}{\varepsilon_{11}}\frac{\partial w}{\partial x} + e_{14}\frac{\partial\psi}{\partial x}, \tag{16}$$

$$D_x = -\varepsilon_{11}\frac{\partial\psi}{\partial x} + e_{14}\frac{\partial w}{\partial y}, \tag{17}$$

$$D_y = -\varepsilon_{11}\frac{\partial\psi}{\partial y} - e_{14}\frac{\partial w}{\partial x}, \tag{18}$$

where the electrically stiffened elastic constant and the slowness of shear bulk waves are given below:

$$\bar{c}_{44} = c_{44} + e_{15}^2/\varepsilon_{11}, \quad s_{sh} = 1/v_{sh} = \sqrt{\rho/\bar{c}_{44}}. \tag{19}$$

3. Propagation of anti-plane surface waves

Surface waves (i.e., waves localized in the region of the gap and vanishing away from it) will be identified in this section for each region of the structure under consideration (Fig. 1). Superscripts and subscripts "*A*", "*B*" and "*C*" will be used to refer to the piezoelectric materials in the regions: $y > a$ and $y < 0$ and to dielectric gap in the region: $0 < y < a$, respectively. The short notations: $c_i = c_{44}^i,\ \bar{c}_i = \bar{c}_{44}^i,\ \varepsilon_i = \varepsilon_{11}^i,\ e_i = e_{15}^i,\ d_i = e_{14}^i,\ \sigma_i = \sigma_{32}^i,\ D_i = D_2^i,\ s_i = s_{sh}^i,\ v_i = v_{sh}^i$ where $i = A, B$ and $\varepsilon_C = \varepsilon_{11}^C,\ D_C = D_2^C$ will be used throughout the paper. Without loss of generality we assume that the surface waves propagate in the positive direction of the *x*-axis.

3.1. Waves in the piezoelectric half-space $y > a$ (region A)

In the piezoelectric half-space, $y > a$, taking into consideration the conditions of wave localization:

$$w_A, \varphi_A \to 0 \quad \text{when} \quad y \to +\infty \tag{20}$$

the surface waves propagating in the *x*-direction can be expressed as follows:

$$w_A = w_{A0} e^{-\xi\lambda_A y} e^{i(\xi x - \omega t)}, \quad \psi_A = \psi_{A0} e^{-\xi y} e^{i(\xi x - \omega t)}, \quad y > a, \tag{21}$$

where w_{A0} and ψ_{A0} are undetermined constants, $\xi > 0$ and $\omega > 0$ are the wave number and frequency.

Substituting Eq. (21) in (14), the equation for ψ_A is automatically satisfied while that for w_A leads to the following expression for λ_A:

$$\lambda_A = \sqrt{1 - v^2/v_A^2} > 0, \tag{22}$$

where $v = \omega/\xi$ is the surface wave velocity of propagation. The positive quadratic root has been chosen in Eq. (22) to satisfy the decaying condition of Eq. (20).

Eq. (22) sets the following restriction on the surface wave velocity of propagation:

$$0 < v < v_A. \tag{23}$$

Substituting Eq. (21) in Eqs. (13) and (15)–(18), the electric potential, anti-plane shear stress and electric displacement of the surface wave in the piezoelectric half-space *A* are determined:

$$\varphi_A = \left(\psi_{A0} e^{-\xi y} + \frac{e_A}{\varepsilon_A} w_{A0} e^{-\xi\lambda_A y}\right) e^{i(\xi x - \omega t)}, \tag{24}$$

$$\sigma_A = -\xi\left(\left(\bar{c}_A\lambda_A - i\frac{d_A e_A}{\varepsilon_A}\right) e^{-\xi\lambda_A y} w_{A0} + (e_A - i d_A) e^{-\xi y}\psi_{A0}\right) e^{i(\xi x - \omega t)}, \tag{25}$$

$$D_A = \xi(\varepsilon_A e^{-\xi y}\psi_{A0} - i d_A e^{-\xi\lambda_A y} w_{A0}) e^{i(\xi x - \omega t)}. \tag{26}$$

3.2. Waves in the piezoelectric half-space $y < 0$ (region B)

In the piezoelectric half-space $y < 0$ the conditions of wave localization are:

$$w_B, \varphi_B \to 0 \quad \text{when} \quad y \to -\infty \tag{27}$$

and surface waves propagating in the *x*-direction can be expressed by:

$$w_B = w_{B0} e^{\xi\lambda_B y} e^{i(\xi x - \omega t)}, \quad \psi_B = \psi_{B0} e^{\xi y} e^{i(\xi x - \omega t)}, \quad y < 0, \tag{28}$$

where w_{B0} and ψ_{B0} are undetermined constants. ξ and ω are the same as in Eq. (21).

Substituting Eq. (28) in (14) we obtain:

$$\lambda_B = \sqrt{1 - v^2/v_B^2} > 0, \tag{29}$$

where $v = \omega/\xi$ is the surface wave velocity of propagation. The positive quadratic root in Eq. (29) ensures the decay of the field components away from the gap.

Eq. (29) sets the following restriction on the surface wave velocity of propagation:

$$0 < v < v_B. \tag{30}$$

Substituting Eq. (28) in Eqs. (13) and (15)–(18), the electric potential, anti-plane shear stress and electric displacement of the surface wave in the piezoelectric half-space *B* are obtained below:

$$\varphi_B = \left(e^{\xi y}\psi_{B0} + \frac{e_B}{\varepsilon_B} e^{\xi\lambda_B y} w_{B0}\right) e^{i(\xi x - \omega t)}, \tag{31}$$

$$\sigma_B = \xi\left(\left(\bar{c}_B\lambda_B + i\frac{d_B e_B}{\varepsilon_B}\right) e^{\xi\lambda_B y} w_{B0} + (e_B + i d_B) e^{\xi y}\psi_{B0}\right) e^{i(\xi x - \omega t)}, \tag{32}$$

$$D_B = -\xi(\varepsilon_B e^{\xi y}\psi_{B0} + i d_B e^{\xi\lambda_B y} w_{B0}) e^{i(\xi x - \omega t)}. \tag{33}$$

3.3. Waves in the gap $0 < y < a$ (region C)

Only an electrostatic field is present in the gap between the two piezoelectric materials. From the governing equation

$$\Delta\varphi_C = 0 \tag{34}$$

we have the following expression for electrostatic waves propagating in the *x*-direction:

$$\varphi_C = [\varphi_{C0}\sinh(\xi y) + \varphi_{C1}\cosh(\xi y)] e^{i(\xi x - \omega t)}, \quad 0 < y < a, \tag{35}$$

where φ_{C0} and φ_{C1} are undetermined constants.

From the constitutive relations and Eq. (35) the electric displacement in the gap is derived which is:

$$D_C = -\varepsilon_C\xi[\varphi_{C0}\cosh(\xi y) + \varphi_{C1}\sinh(\xi y)] e^{i(\xi x - \omega t)}, \quad 0 < y < a. \tag{36}$$

3.4. Dispersion equation

The general explicit expressions for surface waves in all the three regions derived in the previous sections contain six unknown constants: two in each of regions *A*, *B* and *C*. To obtain the dispersion equation, which will determine the dependence of the surface wave velocities on wave number, the continuity conditions at the interfaces must be satisfied. This gives the following system of six homogeneous algebraic equations:

$$(\bar{c}_A\lambda_A - i d_A e_A/\varepsilon_A) e^{-\xi\lambda_A a} w_{A0} + (e_A - i d_A) e^{-\xi a}\psi_{A0} = 0, \tag{37}$$

$$\psi_{A0} e^{-\xi a} + \frac{e_A}{\varepsilon_A} w_{A0} e^{-\xi\lambda_A a} = \varphi_{C0}\sinh(\xi a) + \varphi_{C1}\cosh(\xi a), \tag{38}$$

$$\varepsilon_A e^{-\xi a}\psi_{A0} - i d_A e^{-\xi\lambda_A a} w_{A0} = -\varepsilon_C[\varphi_{C0}\cosh(\xi a) + \varphi_{C1}\sinh(\xi a)], \tag{39}$$

$$(\bar{c}_B\lambda_B + i d_B e_B/\varepsilon_B) w_{B0} + (e_B + i d_B)\psi_{B0} = 0, \tag{40}$$

$$\psi_{B0} + w_{B0} e_B/\varepsilon_B = \varphi_{C1}, \tag{41}$$

$$\varepsilon_B\psi_{B0} + i d_B w_{B0} = \varepsilon_C\varphi_{C0}. \tag{42}$$

To have a non-zero solution, the system of Eqs. (37)–(42) must have a vanishing determinant of the coefficient matrix. This condition, after some algebraic manipulations, yields the dispersion equation below:

$$F_A(v) + F_B(v) = [1 + F_A(v)F_B(v)]\tanh(a\xi), \quad 0 < v < \min(v_A, v_B), \tag{43}$$

where

$$F_A(v) = \frac{1}{\varepsilon_{rA}}\frac{k_A^2 - \lambda_A(v)}{\gamma_A^2 + \lambda_A(v)}, \quad F_B(v) = \frac{1}{\varepsilon_{rB}}\frac{k_B^2 - \lambda_B(v)}{\gamma_B^2 + \lambda_B(v)}. \tag{44}$$

In Eq. (44) the following electromechanical coupling coefficients:

$$k_A = \frac{e_A}{\sqrt{\varepsilon_A \bar{c}_A}}, \quad \gamma_A = \frac{d_A}{\sqrt{\varepsilon_A \bar{c}_A}}, \quad k_B = \frac{e_B}{\sqrt{\varepsilon_B \bar{c}_B}}, \quad \gamma_B = \frac{d_B}{\sqrt{\varepsilon_B \bar{c}_B}}, \tag{45}$$

and the following ratios of dielectric permittivities:

$$\varepsilon_{rA} = \varepsilon_A/\varepsilon_C, \quad \varepsilon_{rB} = \varepsilon_B/\varepsilon_C \tag{46}$$

have been introduced. The restriction on v in Eq. (43) is due to Eqs. (23) and (30).

We note that the dispersion equation (43) depends on the geometrical properties of the structure only through the relative width of the gap, $\eta = a\xi = 2\pi a/\lambda_0$, where a is the absolute gap width and λ_0 is wavelength (Danoyan and Piliposian, 2007, 2008). Because of this physical nature of the product, $a\xi$, dispersion equation (43) can be considered as an equation where v is the unknown velocity of the surface wave propagation to be determined and $\eta = a\xi$ is a parameter (Danoyan and Piliposian, 2007, 2008; Liu et al., 2003; Li and Yang, 2006).

Once the dispersion relation, Eq. (43), is satisfied, the general solution of the system of Eqs. (37)–(42) can now be obtained. That is,

$$w_{A0} = \alpha_0 e^{\xi\lambda_A a}\bar{c}_B \frac{e_A - id_A}{\varepsilon_A \bar{c}_A \lambda_A + d_A^2} \times \left[\varepsilon_B(\lambda_B + \gamma_B^2)\cosh(\xi a) + \varepsilon_C\left(\lambda_B - k_B^2\right)\sinh(\xi a)\right], \tag{47}$$

$$\psi_{A0} = -\alpha_0 e^{\xi a}\frac{\bar{c}_B}{\varepsilon_A}\frac{\varepsilon_A \bar{c}_A \lambda_A - id_A e_A}{\varepsilon_A \bar{c}_A \lambda_A + d_A^2} \times \left[\varepsilon_B(\lambda_B + \gamma_B^2)\cosh(\xi a) + \varepsilon_C\left(\lambda_B - k_B^2\right)\sinh(\xi a)\right],$$

$$w_{B0} = -\alpha_0(e_B + id_B), \quad \psi_{B0} = \alpha_0 \frac{\varepsilon_B \bar{c}_B \lambda_B + id_B e_B}{\varepsilon_B} \tag{49}$$

$$\varphi_{C0} = \alpha_0 \frac{\varepsilon_B \bar{c}_B}{\varepsilon_C}(\lambda_B + \gamma_B^2), \quad \varphi_{C1} = \alpha_0 \bar{c}_B(\lambda_B - k_B^2). \tag{50}$$

where α_0 is an arbitrary constant.

If all the piezoelectric constants vanish then obviously no shear surface wave can be guided by the structure under consideration so that we will assume that at least one of the piezoelectric constants, e_A, d_A, e_B and d_B, is non-zero.

4. Special cases

4.1. Small gap or large wavelength

4.1.1. Infinitesimal gap or infinitely large wavelength

By letting $a\xi \to 0$, the dispersion equation (43) is simplified to:

$$F_A(v) + F_B(v) = 0, \quad 0 < v < v_A, \tag{51}$$

where due to symmetry, and without any loss of generality, we have assumed $v_A \leqslant v_B$.

This limiting case $a\xi \to 0$ can be achieved in two different ways: either by fixing the value of ξ and let a tend to zero or by fixing the value of a and let ξ approach zero. The first one corresponds to two half-spaces with an infinitesimal gap between them and the second corresponds to surface waves with infinitely large wavelength.

From Eqs. (51) and (A.1)–(A.7), we may draw the following conclusions:

1. Surface waves in this special case are not dispersive.
2. Due to Eqs. (51), (A.1), (A.4) and (A.5), no more than one surface wave can be guided by the structure and its existence condition is:

$$\lim_{v\to v_A} F_A(v) + \lim_{v\to v_A} F_b(v) > 0. \tag{52}$$

The limits, $\lim_{v\to v_A} F_A(v)$ and $\lim_{v\to v_A} F_b(v)$, are calculated from Eqs. (A.2) and (A.3) for all possible parametric values of the problem.
3. The surface wave velocity of propagation v_s satisfies the following inequalities:

$$\min\left(v_{bgA}^{el}, v_{bgB}^{el}\right) \leqslant v_s \leqslant \max\left(v_{bgA}^{el}, v_{bgB}^{el}\right). \tag{53}$$

If $d_A = d_B = 0$, then this special case coincides with the special case $K = 0$ in the paper by Fan et al. (2006) and Eq. (51) is equivalent to Eq. (17) in their paper. The dispersion equation (17) of the paper by Fan et al. (2006) is mentioned to be new but is not analysed. By substituting $d_A = d_B = 0$ in Eqs. (A.2) and (A.3) and using Eq. (52), we obtain the following existence condition for the case of hexagonal piezoelectric materials which complements the results of Fan et al. (2006):

$$e_A \neq 0 \quad \text{or} \quad \left[e_B \neq 0 \ \text{and} \ v_B\left(1 - k_B^4(1 + \varepsilon_B/\varepsilon_A)^{-2}\right)^{1/2} < v_A \leqslant v_B\right]. \tag{54}$$

The inequalities (53) specify the region where the surface wave velocity can vary.

4.1.2. Tiny gap or very large wavelength

In this section we will consider the case when $a\xi$ is very small but not infinitesimally small. In real applications, the gap between the materials can be very small, but as we will show, the dynamic behaviour of even a tiny gap can be significantly different from the behaviour of an infinitesimal gap.

As shown in Section 4.1.1 no more than one surface wave can be guided by the gap when $\eta = a\xi \to 0$. Due to continuity no more than one surface wave will be present if $a\xi$ is very small but not infinitesimal. Denoting that unique surface wave velocity by $v(\eta)$ from Eq. (43) we have:

$$F_A(v(\eta)) + F_B(v(\eta)) = [1 + F_A(v(\eta))F_B(v(\eta))]\tanh\eta. \tag{55}$$

Differentiating Eq. (55), substituting $\eta = 0$ and denoting $v_0 = \lim_{\eta\to 0} v(\eta)$ in view of Eq. (51) we obtain:

$$v'(0) = \frac{1 - [F_A(v_0)]^2}{(dF_A/dv + dF_B/dv)|_{v=v_0}}. \tag{56}$$

Using Eq. (56) we have the following asymptotic expansion of the surface wave velocity for small values of $\eta = a\xi$:

$$v(\eta) = v_0 + \eta\frac{1 - [F_A(v_0)]^2}{(dF_A/dv + dF_B/dv)|_{v=v_0}} + O(\eta^2) \quad \text{when} \quad \eta = a\xi \to 0, \tag{57}$$

where v_0 is the solution of Eq. (51). The derivatives of F_A and F_B are presented in Eqs. (A.4) and (A.5).

Numerical calculations are presented in Section 7, which will show that the presence of the second term in Eq. (57) has significant influence on the velocity of the surface wave. This demonstrates that even a tiny gap between two piezoelectric materials cannot always be modelled as an infinitesimal gap and the dielectric permittivity of the material filling the gap cannot be neglected. Thus, cuts and cracks between piezoelectric materials also cannot always be modelled as infinitesimal.

4.2. Infinitely large gap or infinitesimal wavelength

Let $a\xi \to \infty$, the dispersion equation (43) is simplified to: $(F_A(v)-1)(F_B(v)-1)=0$, which splits into two independent equations:

$$\frac{1}{\varepsilon_{rA}}\frac{k_A^2-\lambda_A(v)}{\gamma_A^2+\lambda_A(v)}=1,\quad \frac{1}{\varepsilon_{rB}}\frac{k_B^2-\lambda_B(v)}{\gamma_B^2+\lambda_B(v)}=1. \tag{58}$$

This limiting case can be achieved either by fixing the wavelength and letting the distance between the half-spaces tend to infinity or by fixing the distance between the half-spaces and letting the wavelength approach zero. Thus, the coupling between the half-spaces vanishes and the following two independent non-dispersive surface waves are guided by the structure:

$$v_{sA}=v_A\sqrt{1-\left(\frac{k_A^2-\varepsilon_{rA}\gamma_A^2}{1+\varepsilon_{rA}}\right)^2}=v_{bgA}^{un},\quad \text{with existence condition}\quad k_A^2>\varepsilon_{rA}\gamma_A^2; \tag{59}$$

$$v_{sB}=v_B\sqrt{1-\left(\frac{k_B^2-\varepsilon_{rB}\gamma_B^2}{1+\varepsilon_{rB}}\right)^2}=v_{bgB}^{un},\quad \text{with existence condition}\quad k_B^2>\varepsilon_{rB}\gamma_B^2. \tag{60}$$

Velocities given in Eqs. (59) and (60) are the Bleustein–Gulyaev (B–G) SAW velocities of propagation for piezoelectric half-spaces with no electrodes.

4.3. Identical piezoelectric half-spaces

In this special case, we have:

$$F_A(v)=F_B(v)=F(v)=\frac{1}{\varepsilon_r}\frac{k^2-\lambda(v)}{\gamma^2+\lambda(v)},\quad v_A=v_B=v_T, \tag{61}$$

and the dispersion relation (43) simplifies to a quadratic equation for $F(v)$:

$$F^2(v)-2F(v)\coth(a\xi)+1=0,\quad 0<v<v_T \tag{62}$$

with solutions:

$$F_1(v)=\coth(a\xi/2),\quad F_2(v)=\tanh(a\xi/2). \tag{63}$$

The corresponding surface wave velocities and existence conditions in view of Eq. (61) are given by:

$$v_1=v_T\sqrt{1-\left(\frac{k^2-\coth(a\xi/2)\varepsilon_r\gamma^2}{1+\varepsilon_r\coth(a\xi/2)}\right)^2},\quad k^2>\coth(a\xi/2)\varepsilon_r\gamma^2; \tag{64}$$

$$v_2=v_T\sqrt{1-\left(\frac{k^2-\tanh(a\xi/2)\varepsilon_r\gamma^2}{1+\varepsilon_r\tanh(a\xi/2)}\right)^2},\quad k^2>\tanh(a\xi/2)\varepsilon_r\gamma^2. \tag{65}$$

These surface waves are dispersive unlike those considered in Sections 4.1, 4.2 and 5.

When $\gamma=0$, Eqs. (64) and (65) recover the already known results (Li and Yang, 2006; Gulyaev and Plesskii, 1977) for the case of piezoelectric materials with hexagonal material symmetry.

4.4. A dielectric half-space

If one of the half-spaces, say the half-space B, is a dielectric, then its electromechanical coupling is equal to zero, i.e., $e_B=0$, $d_B=0$, so that $F_B(v)\equiv -1/\varepsilon_{rB}$ and the dispersion equations (43) and (44) can be solved explicitly. The resulting surface wave velocity is:

$$v=v_A\sqrt{1-\left[\frac{(\varepsilon_{rB}+\tanh(a\xi))k_A^2-\varepsilon_{rA}(\varepsilon_{rB}\tanh(a\xi)+1)\gamma_A^2}{\varepsilon_{rA}+\varepsilon_{rB}+(1+\varepsilon_{rA}\varepsilon_{rB})\tanh(a\xi)}\right]^2}$$

with an existence condition:

$$\frac{(\varepsilon_{rB}+\tanh(a\xi))k_A^2-\varepsilon_{rA}(\varepsilon_{rB}\tanh(a\xi)+1)\gamma_A^2}{\varepsilon_{rA}+\varepsilon_{rB}+(1+\varepsilon_{rA}\varepsilon_{rB})\tanh(a\xi)} > \sqrt{1-\left(\frac{\min(v_A,v_B)}{v_A}\right)^2}.$$

5. Special non-dispersive gap waves

Before considering the general case we study the possibility of existence of non-dispersive gap waves in the structure. As the system of two piezoelectric half-spaces separated by a gap contains a characteristic length (i.e., the gap width) the existence of non-dispersive waves in this structure seems to be unexpected. In this section, we prove the possibility of existence of such non-dispersive gap waves, demonstrate how they can be realized and show that their velocities of propagation do not depend on the distance between the piezoelectric half-spaces. From the mathematical viewpoint we consider such solutions of the dispersive equation, Eq. (43), that make both sides of Eq. (43) equal to zero. For the structure under consideration it is equivalent to considering solutions of Eq. (43) with either $|F_A(v)|=1$ or $|F_B(v)|=1$.

If $F_A(v)=1$, from the dispersion relation (43) it follows that $F_B(v)=-1$. Conversely, if $F_B(v)=-1$, we can obtain $F_A(v)=1$ from Eq. (43). If $F_A(v)=1$ and $F_B(v)=-1$, Eq. (43) is satisfied and with Eq. (44), a single surface wave can be guided by the structure with the following velocity of propagation:

$$v_s=v_A\sqrt{1-\left(\frac{k_A^2-\varepsilon_{rA}\gamma_A^2}{1+\varepsilon_{rA}}\right)^2}, \tag{66}$$

and the following existence conditions:

$$\text{either}\quad v_A^2\left[1-\left(\frac{k_A^2-\varepsilon_{rA}\gamma_A^2}{1+\varepsilon_{rA}}\right)^2\right]=v_B^2\left[1-\left(\frac{k_B^2+\varepsilon_{rB}\gamma_B^2}{1-\varepsilon_{rB}}\right)^2\right],\quad \varepsilon_C>\varepsilon_B,\quad k_A^2>\varepsilon_{rA}\gamma_A^2,\quad k_B^2+\gamma_B^2>0, \tag{67}$$

$$\text{or}\quad \varepsilon_B=\varepsilon_C,\quad k_B=0,\quad \gamma_B=0,\quad k_A^2>\varepsilon_{rA}\gamma_A^2. \tag{68}$$

From Eq. (67), it follows that $k_B^2 + \varepsilon_{rB}\gamma_B^2 < 1 - \varepsilon_{rB}$. Using this inequality and Eq. (46), the existence conditions (67) can be written in the following equivalent form:

$$v_A^2\left[1-\left(\frac{k_A^2-\gamma_A^2\varepsilon_A/\varepsilon_C}{1+\varepsilon_A/\varepsilon_C}\right)^2\right] = v_B^2\left[1-\left(\frac{k_B^2+\gamma_B^2\varepsilon_B/\varepsilon_C}{1-\varepsilon_B/\varepsilon_C}\right)^2\right] \tag{69}$$

and

$$k_A \neq 0; \quad [k_B \neq 0 \quad \text{or} \quad \gamma_B \neq 0]; \quad \varepsilon_C > \varepsilon_{CA}^{\min}, \tag{70}$$

where

$$\varepsilon_{CA}^{\min} = \max\left(\frac{1+\gamma_B^2}{1-k_B^2}\varepsilon_B, \frac{\gamma_A^2}{k_A^2}\varepsilon_A\right). \tag{71}$$

If the existence conditions, Eq. (68), are satisfied, the half-space B loses its piezoelectric properties and becomes a dielectric medium with the same dielectric permittivity as the gap, so that it is not surprising that in this case the surface wave, Eq. (59), guided by a single piezoelectric half-space with no electrode is recovered. This is, however, a trivial case.

Unlike the case of Eq. (68), the surface wave with propagation velocity, Eq. (66), and existence conditions, Eqs. (69), (70), is not a trivial one and has some special unique properties. The following conclusions can be obtained from Eqs. (66) and (69)–(71):

(a) Both piezoelectric half-spaces must preserve an electroelastic coupling: $k_A \neq 0$ and either $k_B \neq 0$ or $\gamma_B \neq 0$.
(b) The piezoelectric materials A and B must be non-identical.
(c) Although the air gap of width a provides a coupling between the piezoelectric half-spaces via the electrostatic field in it, the surface wave velocity of propagation and its existence conditions are both independent of a.
(d) Despite the presence of a geometrical characteristic length in the problem, this surface wave is non-dispersive.
(e) The velocity of propagation of this surface wave is equal to the velocity of propagation of the surface wave, Eq. (59), guided by a single piezoelectric half-space A with no electrode.
(f) From Eqs. (66), (69), (A.6), (A.7) and (A.8) it follows that:
$$v_{bgA}^{el} < v_s < v_{bgB}^{el}. \tag{72}$$
(g) As the left and right-hand side of Eq. (69) are, respectively, monotonic decreasing and monotonic increasing functions of $\varepsilon_C > \varepsilon_{CA}^{\min}$ and tend, respectively, to $(v_{bgA}^{el})^2$ and $(v_{bgB}^{el})^2$ when $\varepsilon_C \to \infty$, the following conditions are necessary and sufficient for the existence of $\varepsilon_C > \varepsilon_{CA}^{\min}$, such that Eq. (69) is satisfied:
$$v_{bgA}^{el} < v_{bgB}^{el},$$
$$v_A^2\left[1-\left(\frac{k_A^2-\gamma_A^2\varepsilon_A/\varepsilon_{CA}^{\min}}{1+\varepsilon_A/\varepsilon_{CA}^{\min}}\right)^2\right] > v_B^2\left[1-\left(\frac{k_B^2+\gamma_B^2\varepsilon_B/\varepsilon_{CA}^{\min}}{1-\varepsilon_B/\varepsilon_{CA}^{\min}}\right)^2\right]. \tag{73}$$
Due to Eqs. (71) and (A.8) the conditions, Eq. (73), are equivalent to the following:
$$v_{bgA}^{el} < v_{bgB}^{el}, \quad \text{if} \quad \frac{1+\gamma_B^2}{1-k_B^2}\varepsilon_B \geqslant \frac{\gamma_A^2}{k_A^2}\varepsilon_A; \tag{74}$$
$$v_B\sqrt{\frac{1-k_B^4}{1-k_A^4}} > v_A > v_B\sqrt{1-\left(\frac{\gamma_A^2\varepsilon_A k_B^2 + k_A^2\gamma_B^2\varepsilon_B}{\gamma_A^2\varepsilon_A - k_A^2\varepsilon_B}\right)^2}, \quad \text{if} \quad \frac{1+\gamma_B^2}{1-k_B^2}\varepsilon_B \leqslant \frac{\gamma_A^2}{k_A^2}\varepsilon_A. \tag{75}$$
(h) From (g) it follows that if the conditions of Eqs. (74), (75) and (70) are satisfied by the parameters of the piezoelectric materials, A and B, then the existence condition, Eq. (69), can be satisfied by an appropriate choice of the dielectric permittivity of the gap. Such a choice exists and is unique.
(i) In the special case of piezoelectric materials with hexagonal symmetry, we have $\gamma_A = \gamma_B = 0$, the conditions, Eqs. (74) and (75), reduce to:
$$v_{bgA}^{el} < v_{bgB}^{el} \tag{76}$$
and Eqs. (66), (69), (70) and (71) become:
$$v_s = v_A\sqrt{1 - k_A^4/(1+\varepsilon_A/\varepsilon_C)^2} \tag{77}$$
and
$$v_A^2\left(1-\frac{k_A^4}{(1+\varepsilon_A/\varepsilon_C)^2}\right) = v_B^2\left(1-\frac{k_B^4}{(1-\varepsilon_B/\varepsilon_C)^2}\right); \quad k_A \neq 0; \quad k_B \neq 0; \quad \varepsilon_C > \frac{\varepsilon_B}{1-k_B^2}. \tag{78}$$
If both piezoelectric half-spaces have non-zero electromechanical coupling coefficients and $v_{bgA}^{el} \neq v_{bgB}^{el}$ then due to symmetry, and without loss of generality, we can assume Eq. (76) is satisfied. Thus, non-dispersive gap waves can propagate between dissimilar hexagonal piezoelectric half-spaces having different B–G SAW velocities and located at a finite distance form each other if the gap is filled with a gas with corresponding dielectric permittivity. The non-dispersive gap wave velocity of propagation depends on the properties of the piezoelectric materials, is independent of the distance between the half-spaces and wave number, and is located between the B–G SAW velocities of the half-spaces.

From Eqs. (66), (69)–(71), (22) and (29), we have:

$$\lambda_A(v_s) = \frac{\varepsilon_C k_A^2 - \gamma_A^2\varepsilon_A}{\varepsilon_C + \varepsilon_A}, \quad \lambda_B(v_s) = \frac{\varepsilon_C k_B^2 + \gamma_B^2\varepsilon_B}{\varepsilon_C - \varepsilon_B}. \tag{79}$$

Substituting Eqs. (79) and (47)–(50) in Eqs. (21), (24)–(26), (28), (31)–(33), (35) and (36), and after some algebraic manipulations, we obtain explicit expressions for the non-dispersive gap wave in different parts of the structure. These results are given in Appendix B.

Similar to the case of $F_A(v) = 1$, the dispersive equation (43) has a solution with $F_A(v) = -1$. In this case the surface wave velocity of propagation is given by:

$$v_s = v_B\sqrt{1-\left[\left(k_B^2 - \varepsilon_{rB}\gamma_B^2\right)/(1+\varepsilon_{rB})\right]^2}, \tag{80}$$

and the non-trivial existence conditions are:

$$v_B^2\left[1-\left(\frac{k_B^2-\gamma_B^2\varepsilon_B/\varepsilon_C}{1+\varepsilon_B/\varepsilon_C}\right)^2\right] = v_A^2\left[1-\left(\frac{k_A^2+\gamma_A^2\varepsilon_A/\varepsilon_C}{1-\varepsilon_A/\varepsilon_C}\right)^2\right], \tag{81}$$

$$k_B \neq 0; \quad [k_A \neq 0 \quad \text{or} \quad \gamma_A \neq 0]; \quad \varepsilon_C > \varepsilon_{CB}^{\min}, \tag{82}$$

where

$$\varepsilon_{CB}^{\min} = \max\left(\frac{1+\gamma_A^2}{1-k_A^2}\varepsilon_A, \frac{\gamma_B^2}{k_B^2}\varepsilon_B\right). \tag{83}$$

Results for this case can be obtained from the corresponding results presented for the case of $F_A(v) = 1$ by interchanging "A" and "B".

The presented non-dispersive surface waves, Eqs. (66), (69)–(71) and (80)–(83), and their unique properties (a)–(i) are believed to be new results obtained in this work which bring new understanding and insights to the problem.

Because the existence conditions in this special case are strongly related to the material properties of the piezoelectric half-spaces and the dielectric gap, it is difficult to satisfy them exactly. Real materials are expected to satisfy these conditions only approximately in which case the waves will not be non-dispersive. Hence, it is important to find out if these dispersive waves will be very close to the non-dispersive ones. This raises a question about the stability of the non-dispersive waves, which is, however, beyond the scope of the present paper and is a topic for future research.

6. The general case

As we have already considered the corresponding special cases in the previous two sections, without any loss of generality, here we will assume that:

$$0 < a\xi < \infty, \quad e_A^2 + d_A^2 \neq 0, \quad e_B^2 + d_B^2 \neq 0, \quad |F_A(v)| \neq 1, \quad |F_B(v)| \neq 1. \tag{84}$$

The analysis of this general case is not straightforward. From Eqs. (84) and (43), it follows that $1 + F_A(v)F_B(v) \neq 0$ so that the dispersion equation, Eq. (43), can be written in the following form:

$$G(v) = \tanh(a\xi), \quad 0 < v < \min(v_A, v_B), \tag{85}$$

where

$$G(v) = \frac{F_A(v) + F_B(v)}{1 + F_A(v)F_B(v)}, \quad 0 < v < \min(v_A, v_B). \tag{86}$$

As $0 < \tanh(a\xi) < 1$, from Eq. (85) we have:

$$0 < \frac{F_A(v) + F_B(v)}{1 + F_A(v)F_B(v)} < 1, \quad 0 < \frac{(F_A(v) - 1)(F_B(v) - 1)}{1 + F_A(v)F_B(v)} < 1 \tag{87}$$

and therefore,

$$(F_A(v) - 1)(F_B(v) - 1)(F_A(v) + F_B(v)) > 0. \tag{88}$$

From Eqs. (87) and (88) it follows that if v satisfies the dispersion equation, Eq. (85), then

$$\text{either} \quad |F_A(v)| < 1, \quad |F_B(v)| < 1; \tag{89}$$
$$\text{or} \quad |F_A(v)| > 1, \quad |F_B(v)| > 1. \tag{90}$$

From Eq. (87) it is clear that $F_A(v)$ and $F_B(v)$ cannot both be negative, so that

$$F_A(v) > 0 \quad \text{or} \quad F_B(v) > 0 \tag{91}$$

and using Eqs. (A.6), (A.7) and (86) we obtain:

$$\min\left(v_{bgA}^{el}, v_{bgB}^{el}\right) < v < \min(v_A, v_B). \tag{92}$$

Calculations show that:

$$\frac{dG(v)}{dv} = \frac{1}{(1 + F_A(v)F_B(v))^2} \times \left[\left(1 - [F_B(v)]^2\right)\frac{dF_A(v)}{dv} + (1 - [F_A(v)]^2)\frac{dF_B(v)}{dv}\right],$$

so that using Eqs. (A.4), (A.5) and (A.9), we have:

$$\frac{dG(v)}{dv} > 0 \quad \text{if} \quad |F_A(v)| < 1, |F_B(v)| < 1;$$
$$\frac{dG(v)}{dv} < 0 \quad \text{if} \quad |F_A(v)| > 1, \quad |F_B(v)| > 1. \tag{93}$$

From the above, we note that the dispersion equation can have solutions only in those sub-intervals of the interval (92) where either Eqs. (89) and (91) or Eqs. (90) and (91) are satisfied. According to Eq. (93) in all these sub-intervals the function $G(v)$ is a monotonous function of its argument v so that the dispersion equation (85) can have no more than one solution in each such sub-interval and the surface wave existence condition is that $G(v) - \tanh(a\xi)$ has different signs at the end-points of the sub-interval.

As from Eqs. (A.4) and (A.5), it follows that both $F_A(v)$ and $F_B(v)$ are monotonic increasing functions of v, an interval satisfying Eqs. (90) and (91) cannot be followed by an interval satisfying Eqs. (89) and (91). Hence, there can be no more than two sub-intervals of the interval specified by Eq. (92) where the conditions of Eqs. (89)–(91) are satisfied. Thus, we conclude that *no more than two shear surface waves can be guided by two piezoelectric materials separated by an air gap.*

To obtain the explicit existence conditions, we first analyse the behaviours of $F_A(v)$ and $F_B(v)$ in the interval of Eq. (92). Calculations show that:

$$\begin{aligned}
&F_j(v) < -1 \text{ if } \min\left(v_{bgA}^{el}, v_{bgB}^{el}\right) < v < v_j\sqrt{1 - \left[\left(k_j^2 + \varepsilon_{rj}\gamma_j^2\right)/(1 - \varepsilon_{rj})\right]^2},\\
&-1 < F_j(v) < 1 \text{ if } v_j\sqrt{1 - \left[\left(k_j^2 + \varepsilon_{rj}\gamma_j^2\right)/(1 - \varepsilon_{rj})\right]^2} < v < v_j\sqrt{1 - \left[(k_j^2 - \varepsilon_{rj}\gamma_j^2)/(1 + \varepsilon_{rj})\right]^2},\\
&F_j(v) > 1 \text{ if } v_j\sqrt{1 - \left[\left(k_j^2 - \varepsilon_{rj}\gamma_j^2\right)/(1 + \varepsilon_{rj})\right]^2} < v < \min(v_A, v_B),
\end{aligned} \tag{94}$$

where $j = A, B$. From Eqs. (89)–(91) and the above analysis, we must have the guided surface waves satisfy one of the following conditions:

(a) $F_A(v) < -1, \ F_B(v) > 1,$
(b) $-1 < F_A(v) < 1, \ -1 < F_B(v) < 1,$
(c) $F_A(v) > 1, \ F_B(v) < -1,$
(d) $F_A(v) > 1, \ F_B(v) > 1.$

No more than one solution can exist for each of these four conditions and no more than two surface waves in total.

For the surface waves satisfying the conditions (a), (b), (c) and (d), we obtain:

The explicit expressions of $v_{a0}, v_{a1}, v_{b0}, v_{b1}, v_{c0}, v_{c1}, v_{d0}$ and v_{d1} are given in Appendix C.

$$\begin{aligned}
&\text{(a)} \quad v_{a0} < v < v_{a1} \quad \text{with existence conditions} \quad v_{a0} < v_{a1}, \quad (G(v_{a0}) - \tanh a\xi)(G(v_{a1}) - \tanh a\xi) < 0,\\
&\text{(b)} \quad v_{b0} < v < v_{b1} \quad \text{with existence conditions} \quad v_{b0} < v_{b1}, \quad (G(v_{b0}) - \tanh a\xi)(G(v_{b1}) - \tanh a\xi) < 0,\\
&\text{(c)} \quad v_{c0} < v < v_{c1} \quad \text{with existence conditions} \quad v_{c0} < v_{c1}, \quad (G(v_{c0}) - \tanh a\xi)(G(v_{c1}) - \tanh a\xi) < 0,\\
&\text{(d)} \quad v_{d0} < v < v_{d1} \quad \text{with existence conditions} \quad v_{d0} < v_{d1}, \quad (G(v_{d0}) - \tanh a\xi)(G(v_{d1}) - \tanh a\xi) < 0.
\end{aligned} \tag{95}$$

If the material constants are such that:

$$[k_A = 0, v_A \leqslant v_{bgB}^{el}] \quad \text{or} \quad [k_B = 0, \quad v_B \leqslant v_{bgA}^{el}] \tag{96}$$

then the interval of Eq. (92) degenerates to an empty set and no surface wave can be guided regardless of the dielectric permittivity of the gap.

7. Numerical results and discussion

Numerical computations are performed in this section. In the following examples, we take $LiIO_3$, TeO_2, PZT-4 and ZnO as materials for piezoelectric half-spaces occupying regions A and B. The permittivity of the dielectric filling region C assumes different values. The classes of symmetry and properties of these materials are listed in Table 1 (Dieulesaint and Royer, 1980; Koos and Wolfe, 1984). From these material constants, the corresponding electrically stiffened elastic constant, electromechanical coupling coefficient, bulk wave velocity and Bleustein–Gulyaev (B–G) surface wave velocity are calculated and shown in Table 2.

Table 1
Material properties and classes of symmetry.

Material	Class	c_{44} (GPa)	e_{14} (C/m^2)	e_{15} (C/m^2)	ε_{11} (nF/m)	ρ (kg/m^3)
$LiIO_3$	6	17.8	0.10	0.89	0.079	5402
TeO_2	422	26.5	0.22	0	2000	6000
PZT-4	6 mm	25.6	0	12.7	6.5	7500
ZnO	6 mm	42.5	0	−0.59	738	5676

Table 2
Electroacoustic constants.

Material	$\bar{c}_{44}$ (GPa)	k	γ	v (m/s)	v_{bg}^{el} (m/s)
$LiIO_3$	27.8	0.60	0.067	2270	2117
TeO_2	26.5	0	0.096	2101	2101
PZT-4	50.4	0.70	0	2593	2257
ZnO	47.2	−0.32	0	2884	2870

From Table 2, $k = 0$ for TeO_2 and its bulk wave velocity is less than the B–G surface wave velocities of $LiIO_3$, PZT-4 and ZnO. Thus, the conditions of Eq. (96) are satisfied for TeO_2/$LiIO_3$, TeO_2/PZT-4 and TeO_2/ZnO pairs, and consequently, no surface wave can be guided by the gap between TeO_2 and $LiIO_3$, TeO_2 and PZT-4, and TeO_2 and ZnO half-spaces regardless of the dielectric permittivity of the gap.

Considering $LiIO_3$ as half-space A and ZnO as half-space B and using Table 2 we check that the conditions (74) are satisfied. Based on the results of Section 5 we conclude that there exists such a value ε_C^0 for the dielectric permittivity of the gap in the case of which a non-dispersive surface wave can be guided by the gap between $LiIO_3$ and ZnO regardless of the fact that the structure of two half-spaces separated by a gap has a characteristic length in it (i.e., the gap width). The value of ε_C^0 is independent of the width of the gap and can be calculated using Eqs. (69)–(71).

We can form 10 different pairs from the four materials listed in Table 1. Calculations based on the results of Section 5 show that non-dispersive surface waves in the presence of a finite gap can be guided only by the following three pairs:

$$\begin{aligned} &LiIO_3 \text{ and ZnO with } \varepsilon_C^0 = 0.088 \text{ nF/m}, \\ &LiIO_3 \text{ and PZT-4 with } \varepsilon_C^0 = 44.28 \text{ nF/m}, \\ &\text{PZT-4 and ZnO with } \varepsilon_C^0 = 0.096 \text{ nF/m}. \end{aligned} \tag{97}$$

Figs. 2 and 3 show the variations of surface waves in $LiIO_3$/ZnO, PZT-4/ZnO and $LiIO_3$/PZT-4 half-spaces *versus* the ratio of the gap width a and wavelength λ. All these plots show that the surface wave velocities are monotonic increasing functions of a/λ when the gap permittivity is less than that corresponding to a non-dispersive surface wave, i.e., $\varepsilon_C < \varepsilon_C^0$. When $\varepsilon_C = \varepsilon_C^0$, the surface wave velocity has a constant value calculated in Section 5. If $\varepsilon_C > \varepsilon_C^0$, then the surface wave velocity decreases with increasing a/λ.

As mentioned above, no surface wave can be guided by the pairs TeO_2/ZnO, TeO_2/PZT-4 and TeO_2/$LiIO_3$. Fig. 2 shows that the pairs $LiIO_3$/ZnO and PZT-4/ZnO are examples of the case when only one surface wave can be guided and the half-spaces $LiIO_3$/PZT-4

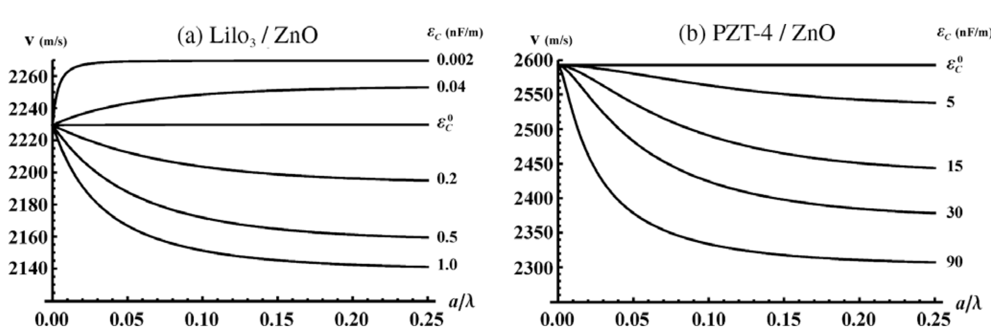

Fig. 2. Gap wave velocities guided by $LiIO_3$ and ZnO, PZT-4 and ZnO half-spaces for different dielectric permittivities of the gap.

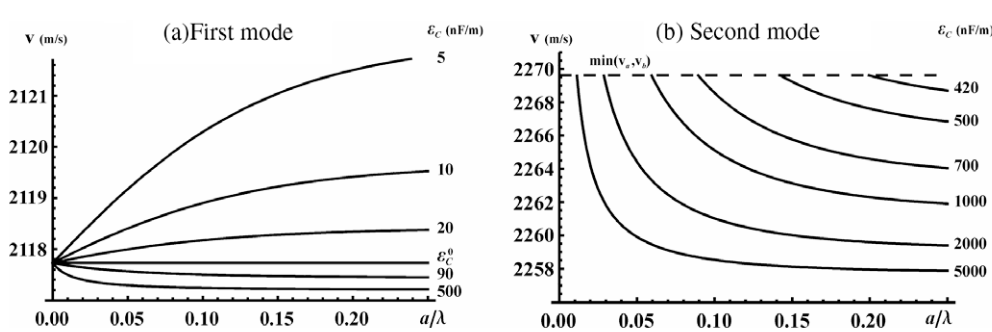

Fig. 3. Gap wave velocities guided by $LiIO_3$ and PZT-4 half-spaces for different dielectric permittivities of the gap.

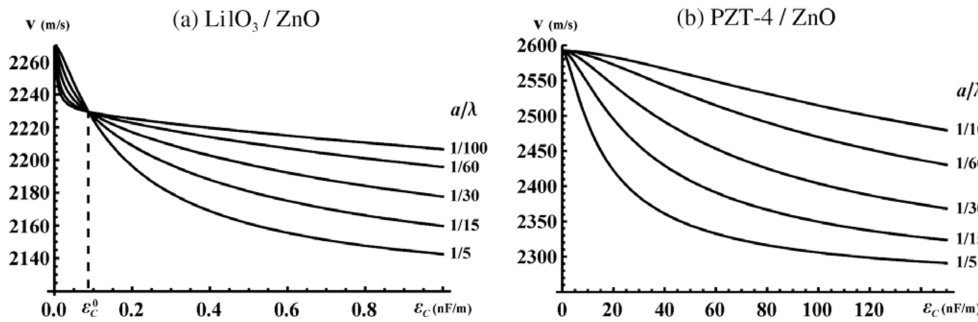

Fig. 4. Variation of surface wave velocities guided by $LiIO_3$/ZnO and PZT-4/ZnO half-spaces for different values of a/λ.

can guide one or two surface waves depending on the permittivity of the gap as demonstrated in Fig. 3. From Section 6 we know that the maximum number of surface waves that can be guided by this structure is equal to 2.

Fig. 4 shows variations of surface wave velocities *versus* gap permittivity. Fig. 4a and b demonstrates the behaviours for relatively smaller and greater values of ε_C. The intersecting point of the curves that is observed in Fig. 4a corresponds to the permittivity $\varepsilon_C = \varepsilon_C^0$, in the case of which the surface wave becomes non-dispersive and independent of the gap width and wavelength. The plots also show that these surface wave velocities monotonically decrease with increase of ε_C.

Now consider the case of a tiny gap between piezoelectric half-spaces studied in Section 4.1.2. The plots in Figs. 2 and 3 show that surface wave velocities have significant variations when a/λ is small and become almost constant as a/λ tends to infinity. Thus, large gaps can be well approximated by infinite gaps and the results of Section 4.2 can be used. At the same time, surface wave velocities are very sensitive to variations of a/λ when a/λ is small, so that even tiny gaps cannot be directly simplified and modelled as just cuts with an infinitesimal width. Fig. 4 demonstrates the variations of surface wave velocity *versus* ε_C from which we can see that the influence of the dielectric permittivity of the gap also cannot be neglected even for the cases of very tiny gaps. From these results, we conclude that both the dielectric permittivity of the dielectric material occupying the gap and the gap width have significant effects on the dynamical properties of the structure and they must be taken into consideration even when the gap width is small compared to the wavelength.

8. Conclusions

Shear gap waves guided at the interface of dissimilar piezoelectric half-spaces are studied. Piezoelectric materials can be dissimilar and belong to different classes of symmetry. The gap wave is mainly concentrated at the interface of the piezoelectric half-spaces and decays exponentially away from it. Simpler special cases when the guided wave velocities can be obtained in closed form are discussed separately. It is shown that in the most general case no more than two surface waves can be guided by this structure under consideration. Explicit existence conditions are derived for all the possible configurations of the structure. Numerical analysis for materials belonging to different classes of symmetry is presented.

It is shown that despite the presence of a characteristic length in the geometry of the problem (i.e., the gap width) a non-dispersive surface wave can be guided by the gap between dissimilar piezoelectric materials if the dielectric permittivity of the material occupying the gap is suitably chosen. An explicit equation with its existence conditions is derived to calculate the value of the gap permittivity which makes it possible to have a non-dispersive gap wave. All the electroelastic wave components of this non-dispersive surface wave are given in explicit form.

The dependence of gap wave velocities on the gap width and permittivity of the material occupying the gap are analysed in detail. It is shown that surface wave velocities are very sensitive to changes of the gap width when it is small and become almost constant when the gap width becomes very large. It is proven that in dynamic problems even a small gap between the piezoelectric materials cannot always be directly simplified and treated as infinitesimal. The dielectric properties of the gap also have significant effects on the surface waves so that small gaps also cannot always be modelled as ideally permeable, impermeable or absorbent.

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Appendix A. Some properties of functions $F_A(v)$ and $F_B(v)$

$$F_A(0) + F_B(0) = \frac{k_A^2 - 1}{\varepsilon_{rA}(\gamma_A^2 + 1)} + \frac{k_B^2 - 1}{\varepsilon_{rB}(\gamma_B^2 + 1)} < 0, \tag{A.1}$$

$$\lim_{v \to v_A} F_A(v) = \begin{cases} \frac{1}{\varepsilon_{rA}} \frac{k_A^2}{\gamma_A^2}, & \text{if } d_A \neq 0, \\ +\infty, & \text{if} d_A = 0,\ e_A \neq 0, \\ -\frac{1}{\varepsilon_{rA}}, & \text{if } d_A = 0,\ e_A = 0, \end{cases} \tag{A.2}$$

$$\lim_{v \to v_A} F_B(v) = \begin{cases} \frac{1}{\varepsilon_{rB}} \left(\frac{k_B^2 + \gamma_B^2}{\gamma_B^2 + \lambda_B(v_A)} - 1 \right), & \text{if } v_A < v_B \text{ or } d_B \neq 0, \\ +\infty, & \text{if } v_A = v_B,\ d_B = 0,\ e_B \neq 0, \\ -\frac{1}{\varepsilon_{rB}}, & \text{if } v_A = v_B,\ d_B = 0,\ e_B = 0, \end{cases} \tag{A.3}$$

$$\frac{dF_A(v)}{dv} = \begin{cases} \frac{1}{\varepsilon_{rA}} \frac{k_A^2 + \gamma_A^2}{[\gamma_A^2 + \lambda_A(v)]^2} \frac{1}{\lambda_A(v)} \frac{v}{v_A^2} > 0, & \text{if } e_A \neq 0 \text{ or } d_A \neq 0, \\ 0, & \text{if } e_A = 0,\ d_A = 0, \end{cases} \quad 0 < v < v_A, \tag{A.4}$$

$$\frac{dF_B(v)}{dv} = \begin{cases} \frac{1}{\varepsilon_{rB}} \frac{k_B^2+\gamma_B^2}{[\gamma_B^2+\lambda_B(v)]^2} \frac{1}{\lambda_B(v)} \frac{v}{v_B^2} > 0, & \text{if } e_B \neq 0 \text{ or } d_B \neq 0, \\ 0, & \text{if } e_B = 0, \ d_B = 0, \end{cases} \quad 0 < v < v_B, \tag{A.5}$$

$$F_A(v) < 0 \text{ if } 0 < v < v_{bgA}^{el}, \quad F_A(v) > 0 \text{ if } v_{bgA}^{el} < v < v_A, \tag{A.6}$$

$$F_B(v) < 0 \text{ if } 0 < v < v_{bgB}^{el}, \quad F_B(v) > 0 \text{ if } v_{bgB}^{el} < v < v_B, \tag{A.7}$$

where

$$v_{bgA}^{el} = v_A\sqrt{1-k_A^4}, \quad v_{bgB}^{el} = v_B\sqrt{1-k_B^4} \tag{A.8}$$

are the Bleustein–Gulyaev (B–G) surface acoustic wave velocities of propagation in the piezoelectric half-spaces if they would have grounded electrode covers.

From Eqs. (A.4) and (A.5) we obtain:

$$\left(\frac{dF_A(v)}{dv}\right)^2 + \left(\frac{dF_B(v)}{dv}\right)^2 > 0 \quad \text{for} \quad 0 < v < \min(v_A, v_B). \tag{A.9}$$

Appendix B. Explicit expressions of the electroelastic non-dispersive surface wave

Piezoelectric region A ($y > a$):

$$w_A = \alpha_0 e^{\xi a} \frac{\varepsilon_C + \varepsilon_A}{\varepsilon_C - \varepsilon_B} \frac{e_B^2 + d_B^2}{e_A + id_A} e^{-(y-a)\xi\lambda_A} e^{i(\xi x - \omega t)},$$

$$\psi_A = -\frac{\alpha_0 e^{\xi a}}{\varepsilon_A} \frac{\varepsilon_C e_A - i\varepsilon_A d_A}{\varepsilon_C - \varepsilon_B} \frac{e_B^2 + d_B^2}{e_A + id_A} e^{-(y-a)\xi} e^{i(\xi x - \omega t)},$$

$$\varphi_A = \frac{\alpha_0 e^{\xi a}}{\varepsilon_A} \frac{e_B^2 + d_B^2}{e_A + id_A} \frac{e_A(\varepsilon_C + \varepsilon_A) e^{-(y-a)\xi\lambda_A} - (\varepsilon_C e_A - i\varepsilon_A d_A) e^{-(y-a)\xi}}{\varepsilon_C - \varepsilon_B} e^{i(\xi x - \omega t)},$$

$$\sigma_A = -\xi\alpha_0 e^{\xi a} \frac{e_B^2 + d_B^2}{\varepsilon_A} \frac{e_A - id_A}{e_A + id_A} \frac{\varepsilon_C e_A - i\varepsilon_A d_A}{\varepsilon_C - \varepsilon_B} \left(e^{-(y-a)\xi\lambda_A} - e^{-(y-a)\xi}\right) e^{i(\xi x - \omega t)},$$

$$D_A = -\xi\alpha_0 e^{\xi a} \frac{e_B^2 + d_B^2}{e_A + id_A} \times \frac{id_A(\varepsilon_C + \varepsilon_A) e^{-(y-a)\xi\lambda_A} + (\varepsilon_C e_A - i\varepsilon_A d_A) e^{-(y-a)\xi}}{\varepsilon_C - \varepsilon_B} e^{i(\xi x - \omega t)}.$$

Piezoelectric region B ($y < 0$):

$$w_B = -\alpha_0 (e_B + id_B) e^{\xi\lambda_B y} e^{i(\xi x - \omega t)},$$

$$\psi_B = \alpha_0 \frac{e_B + id_B}{\varepsilon_B} \frac{\varepsilon_C e_B - i\varepsilon_B d_B}{\varepsilon_C - \varepsilon_B} e^{\xi y} e^{i(\xi x - \omega t)},$$

$$\varphi_B = \alpha_0 \frac{e_B + id_B}{\varepsilon_B} \left(\frac{\varepsilon_C e_B - i\varepsilon_B d_B}{\varepsilon_C - \varepsilon_B} e^{\xi y} - e_B e^{\xi\lambda_B y}\right) e^{i(\xi x - \omega t)},$$

$$\sigma_B = -\frac{\alpha_0 \xi}{\varepsilon_B} \frac{\varepsilon_C e_B - i\varepsilon_B d_B}{\varepsilon_C - \varepsilon_B} (e_B + id_B)^2 \left(e^{\xi\lambda_B y} - e^{\xi y}\right) e^{i(\xi x - \omega t)},$$

$$D_B = \xi\alpha_0 (e_B + id_B) \left(id_B e^{\xi\lambda_B y} - \frac{\varepsilon_C e_B - i\varepsilon_B d_B}{\varepsilon_C - \varepsilon_B} e^{\xi y}\right) e^{i(\xi x - \omega t)}.$$

Gap region C ($0 < y < a$):

$$\varphi_C = \alpha_0 \frac{e_B^2 + d_B^2}{\varepsilon_C - \varepsilon_B} e^{\xi y} e^{i(\xi x - \omega t)}, \quad D_C = -\varepsilon_C \xi \alpha_0 \frac{e_B^2 + d_B^2}{\varepsilon_C - \varepsilon_B} e^{\xi y} e^{i(\xi x - \omega t)}.$$

Appendix C. Explicit expressions for some main velocities

$$v_{a0} = \max\left(\min\left(v_{bgA}^{el}, v_{bgB}^{el}\right), v_B\sqrt{1 - \left[\left(k_B^2 - \varepsilon_{rB}\gamma_B^2\right)/(1+\varepsilon_{rB})\right]^2}\right),$$

$$v_{a1} = \min\left(v_A\sqrt{1 - \left[\left(k_A^2 + \varepsilon_{rA}\gamma_A^2\right)/(1-\varepsilon_{rA})\right]^2}, v_B\right),$$

$$v_{b0} = \max\left(v_A\sqrt{1 - [(k_A^2 + \varepsilon_{rA}\gamma_A^2)/(1-\varepsilon_{rA})]^2}, v_B\sqrt{1 - \left[\left(k_B^2 + \varepsilon_{rB}\gamma_B^2\right)/(1-\varepsilon_{rB})\right]^2}\right),$$

$$v_{b1} = \min\left(v_A\sqrt{1 - [(k_A^2 - \varepsilon_{rA}\gamma_A^2)/(1+\varepsilon_{rA})]^2}, v_B\sqrt{1 - \left[\left(k_B^2 - \varepsilon_{rB}\gamma_B^2\right)/(1+\varepsilon_{rB})\right]^2}\right),$$

$$v_{c0} = \max\left(\min\left(v_{bgA}^{el}, v_{bgB}^{el}\right), v_A\sqrt{1 - \left[\left(k_A^2 - \varepsilon_{rA}\gamma_A^2\right)/(1+\varepsilon_{rA})\right]^2}\right),$$

$$v_{c1} = \min\left(v_B\sqrt{1 - \left[\left(k_B^2 + \varepsilon_{rB}\gamma_B^2\right)/(1-\varepsilon_{rB})\right]^2}, v_A\right),$$

$$v_{d0} = \max\left(v_A\sqrt{1 - [(k_A^2 - \varepsilon_{rA}\gamma_A^2)/(1+\varepsilon_{rA})]^2}, v_B\sqrt{1 - \left[\left(k_B^2 - \varepsilon_{rB}\gamma_B^2\right)/(1+\varepsilon_{rB})\right]^2}\right),$$

$$v_{d1} = \min(v_A, v_B).$$

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